Unpacking mathematical imagination

Spacchettando l'immaginazione matematica

Desempacando la imaginación matemática

Panayiota Irakleous, Constantinos Christou
and
Demetra Pitta-Pantazi
University of Cyprus, Cipro

Abstract. *The current study aims to empirically examine the structure of mathematical imagination. A mathematical imagination test was administered to 217 sixth-grade students from three urban and eight rural primary schools. Partial Least Squares structural equation modeling (PLS-SEM) was employed through Smart PLS in order to empirically examine the proposed model. The data analysis yields that the proposed model met all evaluation criteria of PLS-SEM and, hence, mathematical imagination can be described in terms of vividness, transformative abilities, and originality. Potential research directions are suggested, and theoretical, methodological, and practical implications are discussed.*

Keywords: mathematical imagination; vividness; transformative abilities; originality.

Sunto. *La presente ricerca ha l'obiettivo di esaminare empiricamente la struttura dell'immaginazione matematica. Un test di immaginazione matematica è stato somministrato a 217 studenti di grado sei di tre scuole primarie cittadine e di otto scuole primarie rurali. Per esaminare empiricamente il modello proposto è stata utilizzata la modellizzazione di equazioni strutturali basate sui minimi quadrati parziali (PLS-SEM) tramite Smart PLS. L'analisi dei dati ha dimostrato che il modello proposto soddisfa tutti i criteri di valutazione del PLS-SEM e, pertanto, l'immaginazione matematica può essere descritta in termini di vividezza, capacità di trasformazione e originalità. Vengono suggerite potenziali direzioni di ricerca e discusse le implicazioni teoriche, metodologiche e pratiche.*

Keywords: immaginazione matematica; vividezza; abilità trasformative; originalità.

Resumen. *La presente investigación tiene como objetivo examinar empíricamente la* *estructura de la imaginación matemática. Se administró una prueba de imaginación matemática a 217 estudiantes de sexto grado de tres escuelas primarias de la ciudad y ocho escuelas primarias de zonas rurales. Para examinar empíricamente el modelo propuesto, se utilizó una modelización de ecuaciones estructurales basado en los mínimos cuadrados parciales (PLS-SEM) a través de Smart PLS. El análisis de los datos demostró que el modelo propuesto cumple con todos los criterios de evaluación del PLS-SEM y, por tanto, la imaginación matemática puede describirse en términos de viveza, capacidad transformadora y originalidad. Se sugieren posibles direcciones de investigación y se discuten las implicaciones teóricas, metodológicas y prácticas.*

Parablas clave. imaginación matemática; viveza; habilidades transformadoras; originalidad.

1. Introduction

Imagination is delineated as a “vital element of mathematical thinking” (Pound & Lee, 2015, p. 5) and “the source of invention, novelty, and generativity” (Egan & Judson, 2016, p. 4). It “acts as the catalyst for all creative actions” (Eckhoff & Urbach, 2008, p. 180) and can be regarded as “the corner stone of creativity” (Christou, 2017, p. 14).

In detail, imagination is characterized as “the driving force behind human thought” (Ho et al., 2013, p. 68). Einstein maintains that imagination is more important than knowledge (Ho et al., 2013). It is a core component of knowledge construction (Lev-Zamir & Leikin, 2011) and transforms knowledge into new ideas (Seelig, 2012). In addition, imagination stimulates problem solving (Lindstrand, 2010). Imagination helps children become creative thinkers and solve difficult problems in new and innovative ways (Eckhoff & Urbach, 2008). It enables learners to unlock and explore mathematical ideas (Jagals & van der Walt, 2018). Moreover, in the early school years, imagination has been linked to children’s more sophisticated cognitive abilities and improved ability to control their emotions (Smith & Mathur, 2009).

Thus, several scholars point to the need to nurture imagination through education. According to Wu and Albanese (2013), society should examine and implement various approaches to allow education to return to imagination, which is the source and destination of knowledge. Imagination should be developed at any time and in all curriculum areas to enrich students’ learning more effectively (Egan & Judson, 2016). If imagination is promoted it is possible that we can all become more creative than we were as children (Eckhoff & Urbach, 2008). Further, engaging children in imaginative activities can improve their enjoyment and learning (Smith & Mathur, 2009).

Notwithstanding the acknowledgment of the importance of imagination for student mathematical learning, the research community has not shed light on imagination through empirical studies (Egan, 2015). It is notable that compared to research on creativity, literature on imagination is less advanced because of vague research questions (Ren et al., 2012). In addition, the definition of imagination remains vague (Ho et al., 2013), given that the field still lacks established theoretical frameworks on imagination (Abrahamson, 2006; Dziedziewicz & Karwowski, 2015; Egan, 1992). The present study seeks to empirically examine the structure of imagination in the domain of mathematics.

1.1. Theoretical perspectives

Imagination is a fuzzy concept (Egan, 1992; van Alphen, 2011), as it can refer to different things (Macknight, 2009). For instance, Seelig (2012) defines imagination as one's ability to create something new. Lothane (2007) asserts that imagination is the basic ability to imagine, visualize, represent all that is experienced through either an image or a word. In addition, according to Pelaprat and Cole (2011), imagination is the process that makes it possible for an individual to emerge and, for the world to come into view. Ho et al. (2013) conceptualize imagination as the ability to construct images in the brain that are further visualised to generate ideas that can solve problems.

The underpinning framework for conceptualizing imagination in the current study is the 'Conjunctional Model of Creative Imaging Ability' (Dziedziewicz & Karwowski, 2015). We purposively build on this theoretical model, because it is a broadly-acknowledged model of imagination derived from the area of psychology. Moreover, this model clearly specifies a set of constructs that can be easily adapted to the field of mathematics.

Based on the 'Conjunctional Model of Creative Imaging Ability', imagination is composed of three constructs: vividness, transformative abilities, and originality (Dziedziewicz & Karwowski, 2015). In this model, vividness is defined as the ability to create lucid and expressive images characterized by high complexity and level of detail. Transformative abilities are the abilities to transform created imageries and finally, originality is the ability to produce creative unique imageries.

In the present paper, we conceptualized and adapted all three constructs to the domain of mathematics. First, we conceive vividness as an aspect related to visualization. Visualization plays a key role not only in geometry learning (Presmeg, 1997), but in algebra as well, as it is described as a vehicle for effective problem solving in algebra (Yerushalmy et al., 1999). Therefore, we link vividness to both spatial and algebraic images. Spatial images are mental constructs which represent spatial information (Presmeg, 1986), while vividness of spatial images is the mental manipulation of spatial objects in various ways (Presmeg, 1997). Drawing on Presmeg's definition (1986) of spatial images, algebraic images can be defined as mental constructs representing algebraic information. Vividness of algebraic images refers to solving algebraic insight problems (Weisberg, 1995), which require students to pass through the four stages of the creative process (preparation, incubation, illumination, and verification) (Wallas, 1926) to reach a solution. According to Vale and Barbosa (2018), "seeing is related to having creative insights or aha moments" (p. 26).

In the stage of preparation, one is motivated to explore the problem situation and collect the necessary information to reach a solution (Yaftian, 2015). This phase involves intentional and conscious work (Liljedhal, 2013). When the solver is unable to reach a solution at a conscious level (Liljedhal, 2004, 2013), the incubation phase begins (Liljedhal, 2004, 2013; Smith, 1995; Yaftian, 2015). The solver may recognize this as an impasse (Savic, 2016), forget the problem for a time, and focus on other activities (Yaftian, 2015). The problem is internalized in the unconscious mind, which continues to process the information (Yaftian, 2015). This phase can last from several minutes to several years (Aldous, 2007; Liljedhal, 2004).

In the illumination phase, the sudden feeling of reaching the problem solution is often accompanied by a sub-vocal or exuberantly shouted Aha! (Webb et al., 2018) and hence is known as an Aha! or Eureka experience (Aldous, 2007; Liljedahl, 2004, 2005, 2013; Shen et al., 2013; Sriraman, 2009; Weisberg, 2015).

The fourth and final stage is called verification and involves "examining, improving, assessing, validating, writing out, controlling, persuading and lastly publishing the new idea" (Yaftian, 2015, p. 2522). The solution is checked and further refined (Aldous, 2007; Haylock, 1987). The solver makes the result precise, searches for possible extensions through utilization of the result and expresses the result in language (Sriraman, 2004). If the verification phase indicates that the solution is not suitable, then the solver may go back to an earlier stage of the problem solving process (Aldous, 2007).

Regarding transformative abilities, Dziedziewicz and Karwowski (2015) define them as the abilities to transform created imageries. In this paper, our definition in the field of mathematics was based on the concept of mathematization, which derives from the theory of realistic mathematics (Jupri & Drijvers, 2016). Mathematization consists of two distinct types: horizontal and vertical (Treffers, 1987). Horizontal mathematization focuses on the process leading from the world of life to the world of symbols, whereas vertical mathematization refers to the process of moving within the symbolic world (Freudenthal, 1991). This study focuses exclusively on horizontal mathematization due to the age of the participants, considering that younger children concretely experience algebra using concrete and visual objects (Lee et al., 2016).

Finally, given that originality is the ability to produce creative unique imageries (Dziedziewicz & Karwowski, 2015), in this paper we defined originality as the "statistical infrequency of the responses in relation to the peer group" (Haylock, 1997, p. 71) and the likelihood of holding new and unique ideas (Gil et al., 2007). In addition, we assume that originality can manifest in students' mathematical products related to vividness and products related to transformative abilities. First, the close relationship between originality and imagination is evident in the literature. For example, Egan (2005) states that imagination is the source of flexibility and originality of human thinking, while White (1990) argues that imagination is tightly connected to invention and originality. In addition, originality is evident in the questions individuals pose, the representations they use, and the justifications they offer (Sheffield, 2009).

The goal of the present paper is to empirically examine the structure of mathematical imagination. The research question of the study is the following: Do the data of the study confirm the structure of the proposed model about mathematical imagination?

2. Methods

2.1. Participants

The participants were 217 sixth-grade primary school students (94 boys, 100 girls, and 23 students whose gender data was missing). Students were selected through convenient sampling and came from 3 urban and 8 rural schools. We focused on primary school students because the basics of creative thinking are developed at early ages (Leikin & Pitta-Pantazi, 2013). We also decided to examine 11-year-old students in the light of related literature. In fact, according to Hennesey (2007), second graders are creative and full of enthusiasm for learning whereas three years later they become passive learners without curiosity. In addition, research data show that 81% of fourth-graders in USA have positive attitudes towards mathematics while four years later only 35% of those students exhibit the same positive attitudes (U.S. Department of Education & National Center for Education Statistics, 2003). Therefore, it seems that the age of 11 years is of particular research interest.

2.2. Data Collection

A mathematical imagination test composed of three parts was administered to students. The tasks were designed based on the Conjunctional Model of Creative Ability (Dziedziewicz & Karwowski, 2015), according to which imagination is the conjunction of vividness, transformative abilities, and originality. Figure 1 presents an example task for each part of the test. Part A consists of three multiple-solution tasks measuring the vividness of spatial images (Levav-Waynberg & Leikin, 2012). Part B focuses on the vividness of algebraic images through three insight problems (Weisberg, 1995), since seeing is associated with having creative insights or ‘Aha!’ moments (Vale & Barbosa, 2018). Insight problems require students to pass through the four stages of the creative process (preparation, incubation, illumination, and verification) (Wallas, 1926) to reach a solution. Lastly, Part C includes three multiple-solution tasks examining transformative abilities and focusing on horizontal mathematization (Freudenthal, 1991).

Figure 1
Example tasks of the mathematical imagination test

Part A

1

Maria collected cherries, as shown in the figure below.

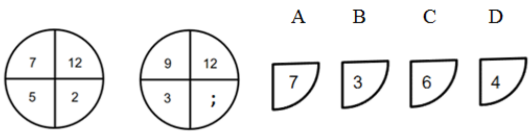
(a) Can you discover three different ways to quickly count them? Write down your calculations.

(b) Can you think of a way that no one else could think of? Write down your calculations.

Part B

1

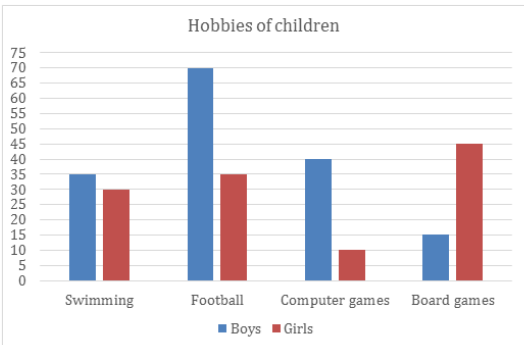
Choose the quarter that is missing from the circle. Explain your answer.



Part C

1

(a) Write three different questions that can be answered based on the information given in the chart below.



(b) Write a question that no one else could pose.

2.3. Scoring Process

The assessment of the vividness of spatial images was based on students' flexibility scores in the three tasks of Part A (indicators: Flex_VSI_1, Flex_VSI_2, Flex_VSI_3), given that imagination is the source of flexibility of human thinking (Egan, 2005). Flexibility pertains to the generation of divergent strategies or solutions in a task (Leikin, 2009). Specifically, we tracked all students' answers in each task and classified them into categories and sub-categories by taking into account the diversity of their answers and their cognitive complexity. For instance, the categories established for the first task of part A were: 'counting', 'serial strategies', 'similar groups and remainder', 'subitizing'. The 'serial strategies' category was sub-divided into 9 sub-categories: vertical series, horizontal series, diagonal series, zig-zag pattern, combination of two 'serial strategies', combination of a 'serial strategy' with a 'similar group' strategy, horizontal groups, vertical groups (of 5 or 8), vertical groups (of 5 or 8) and then subtraction. Students' flexibility was evaluated using the scoring scheme in Table 1, which draws on the scoring scheme proposed by Leikin (2013).

Figure 2 presents an example of a student's answer given for the first task of part A. Specifically, 1 point was given for the first correct solution. The second solution was given 1 point, as it belongs to a category of answers different from the answer(s) given previously (serial strategy). The third solution was attributed 0.1 points respectively, as it belongs to the previous category of answers (serial strategy), but to a different sub-category (horizontal series). Finally, 1 point was given for the fourth solution, as it belongs to a category of answers different from the answer(s) given previously ('subitizing' category). Overall, the student's total score was 3.1 points.

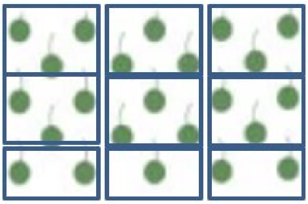
Table 1
Scoring scheme for evaluating flexibility

	Points per solution
For the first correct solution	1
which belongs to a category of answers different from the answer(s) given previously	1
which belongs to one of the previous categories of answers, but to a different sub-category	0.1
which belongs to one of the previously used categories and sub-categories of answers	0
Incorrect or no answer	0

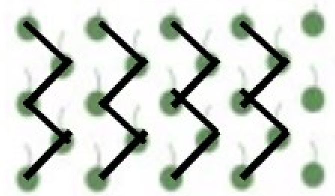
Figure 2

Indicative answers given to the first task of part A in the mathematical imagination test

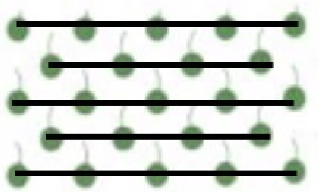
1st solution: 'Similar groups' category
Groups of three



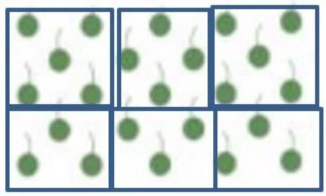
2nd solution: Serial strategies
Zig-zag pattern



3rd solution: Serial strategies
Horizontal series



4th solution: Subitizing
Dice pattern



Students' vividness of algebraic images was assessed on the basis of the correctness of their answers in the insight problems of Part B (indicators: VAI_1, VAI_2, VAI_3). In these insight problems, students should pass through the four stages of creative process (preparation, incubation, illumination, and verification) (Wallas, 1926) to reach the correct solution. In order to ascertain whether students had an 'Aha!' experience, while solving the three insight problems or not, we considered both the correctness of the answer and the justifications provided. It is worth mentioning that we purposefully used problems with solutions for which only one constraint needs to be relaxed, as those problems facilitate the examination of Aha! experiences (Danek & Wiley, 2017). Each correct answer and each correct justification were given 1 point respectively. Yet, half points were given to partially correct answers and justifications.

With respect to students' transformative abilities, for each student, we calculated flexibility scores in the three tasks of Part C (indicators: Flex_TA_1, Flex_TA_2, Flex_TA_3). In short, we established categories and sub-categories of students' responses. Students' responses in the first task of Part C were grouped into five categories: 1) obvious answers, 2) comparison of children, 3) sum of children, 4) fractions-percentages-ratios and 5) adding or extending assumptions. For instance, category 3 was analyzed into three sub-categories:

total number of students for a single activity, total number of boys or girls who prefer 2 or 3 activities, and total number of boys or girls or children. Flexibility scores for each task were calculated, according to the scoring scheme shown in Table 1.

For example, the questions posed by a student were as follows: How many boys like swimming? (obvious answers), How many girls like board games? (obvious answers), How many kids like football? (sum of children), How many more boys like football than girls? (comparison of children). In fact, 1 point was given for the first correct solution. The second solution was attributed 0 points, as it belongs to the previous category and sub-category of answers. Furthermore, 1 point was given to the third and fourth answer respectively because they both belong to a category of answers different from the answer(s) given previously. Overall, student's total score was 3 points.

Originality was assessed by taking under consideration students' responses in the multiple-solution tasks of the imagination test: the tasks of Part A on the vividness of spatial images (indicators: Or_VSI_1, Or_VSI_2, Or_VSI_3) and tasks of Part C on transformative abilities (Or_TA_1, Or_TA_2, Or_TA_3). While assessing originality, 'relative' and 'absolute' assessment of originality were combined (Leikin, 2009; 2013). Therefore, the assessment of originality was based on two criteria: statistical infrequency of the responses in relation to the peer group and cognitive complexity of the category to which answers belong. Statistical infrequency refers to the relative assessment of originality. First, we divided the frequency of each correct answer by the total number of correct answers provided by the students in the population under study to calculate the percentage frequency of that answer. Each correct answer was given a score between zero and one, according to a scoring rubric. The rarest correct solution received the highest score. As for cognitive complexity, it refers to absolute originality, which is based on the level of insight embedded in the solution process used by students (Ervynck, 1991). Each category of answers in each task was assigned to one of the following three levels of cognitive complexity: low level (0 points given), moderate level (1 point given), and high level (2 points given). Originality scores were calculated as the sum of the score allocated for the statistical infrequency of each response and the score allocated for its cognitive complexity.

The process of scoring the originality of students' answers presented in Figure 2 will be explained below. As mentioned before, the evaluation of originality was based on two criteria: statistical infrequency of the responses in relation to the peer group and cognitive complexity of the category to which answers belong. 'Groups of three' solution was given 0.2 points for statistical infrequency and 1 point for cognitive complexity. 'Zig zag pattern' solution was given 0.6 points for statistical infrequency and 1 point for cognitive complexity.

'Horizontal series' solution was given 0.2 points for statistical infrequency and 1 point for cognitive complexity. Finally, 'Dice pattern' solution was awarded 0.8 points for statistical infrequency and 2 points for cognitive complexity. In total, the student's originality score in that task was 6.8 points.

2.4. Reliability and Validity of the Instruments

As for the reliability of the instrument, internal consistency was found to be moderate to high (Cronbach α=.85) (Murphy & Davidshofer, 2001). The content validity of the instruments was measured as well, by using the Content Validity Index (CVI). Item-level CVI (I-CVI) refers to the content validity of individual items and scale-level CVI (S-CVI) reflects the content validity of the overall scale (Polit & Beck, 2006).

Regarding item-level CVI (I-CVI), we asked four experts (two mathematics primary teachers and two mathematics education scholars) to rate each item of the instruments in terms of its relevance to the underlying construct. Lynn (1986) suggested that the ideal number of experts is three to ten. The experts used a 4-point scale to avoid having a neutral and ambivalent midpoint (Lynn, 1986), where 1 indicated not relevant, 2 for somewhat relevant, 3 for quite relevant, and 4 for highly relevant (Davis, 1992). Then, for each item, the I-CVI was computed as the number of experts giving a rating of either 3 or 4 divided by the total number of experts (Polit & Beck, 2006). When experts are fewer than five, only an I-CVI of 1.00 is acceptable (Lynn, 1986). In other words, all experts must agree on the content validity of each item. All I-CVIs had values of 1.00, showing satisfactory content relevance.

The scale-level CVI (S-CVI) is "the proportion of items on an instrument that achieved a rating of 3 or 4 by the content experts'' (Beck & Gable, 2001, p. 209). An interpretation of the S-CVI definition is S-CVI/Ave. To calculate S-CVI/Ave, we computed the average of the I-CVIs in each scale. S-CVI/Ave values above .90 are regarded as satisfactory (Waltz et al., 2005). All S-CVIs had values of 1.00, illustrating acceptable content relevance.

2.5. Data Analysis

Descriptive statistics were calculated using the SPSS software package, while PLS-Structural Equation Modeling was conducted using the SmartPLS software. The purpose of PLS-SEM is to predict and explain the variance of a target construct (Sarstedt et al., 2017). Specifically, we examined a reflective measurement model. In a reflective measurement model, latent variables are measured using reflective (effect) indicators (Diamantopoulos & Siguaw, 2006).

This method was chosen for two reasons. First, when performing PLS-SEM, researchers benefit from the method's greater statistical power compared to factor-based SEM and, hence, the PLS-SEM method is more likely to identify an effect as significant when it is indeed (Sarstedt et al., 2017). Second, in contrast to factor-based SEM, when applying the PLS-SEM algorithm, the overall number of model parameters can be extremely high in relation to the sample size as long as each partial regression relationship draws on a sufficient number of observations (Sarstedt et al., 2017).

In order to evaluate this type of model, indicators' reliability, internal consistency reliability, convergent validity and discriminant validity should be considered (Sarstedt et al., 2017). Indicators' reliability is assessed using indicators' loadings. Loadings above .70 reveal that the indicator has an acceptable degree of reliability (Sarstedt et al., 2017). Composite reliability ρ_c and Cronbach α are used for examining the constructs' internal consistency. A value of .60 is considered as a threshold for both reliability ρ_c and Cronbach α (Hair et al., 2017). However, values above .95 imply that the items are almost identical. Convergent validity is examined by the average variance extracted (AVE) across all items associated with a particular construct. An acceptable benchmark of AVE is .50 or higher, meaning that, on average, the construct explains (more than) 50% of the variance of its items (Sarstedt et al., 2017). Finally, discriminant validity examination shows the extent to which a construct is empirically distinct from other constructs (Sarstedt et al., 2017). Discriminant validity is evaluated based on the Fornell-Larcker (1981) criterion and the cross-loadings (Chin, 1998), which is also known as 'item-level discriminant validity' (Henseler et al., 2015). The Fornell-Larcker (1981) criterion recommends that discriminant validity of a construct is achieved when the square root of the AVE is greater than the correlation between the constructs of the model. Regarding the item-level discriminant validity, each indicator loading should be greater than all of its cross-loadings (Chin, 1998).

3. Findings

This section begins with a descriptive analysis of the data obtained and then presents the results emerged from the evaluation of the reflective measurement model measuring imagination in mathematics. Table 2 presents the descriptive statistics of the mathematical imagination test.

All variables were converted to a scale of 0-1, to allow comparisons among students' scores in each variable. The mean scores were between .17 and .62. Originality in transformative abilities task 2 had the lowest mean score. This task asked students to pose various problems which could be answered by a given mathematical sentence of additive structure. The highest mean score was achieved in task 2 on vividness of algebraic images. This task called students to find the arithmetic values represented by symbols in a given addition.

Regarding skewness and kurtosis, in large samples it is appropriate to focus on the shape of the distribution instead of using formal inference tests (Tabachnick & Fidell, 2014). Because the standard errors for both skewness and kurtosis decrease with larger N, the null hypothesis is likely to be rejected with large samples when there are only minor deviations from normality. Based on Table 2, the absolute values of skewness indices were below one, except for originality in transformative abilities task 2. In addition, the absolute values of kurtosis indices of all variables were lower than 2 and in some cases close to zero. An exception was observed in the kurtosis index of originality in transformative abilities task 2. These indices recommend that the measure of mathematical imagination was normally distributed.

Table 2
Descriptive statistics of the mathematical imagination test

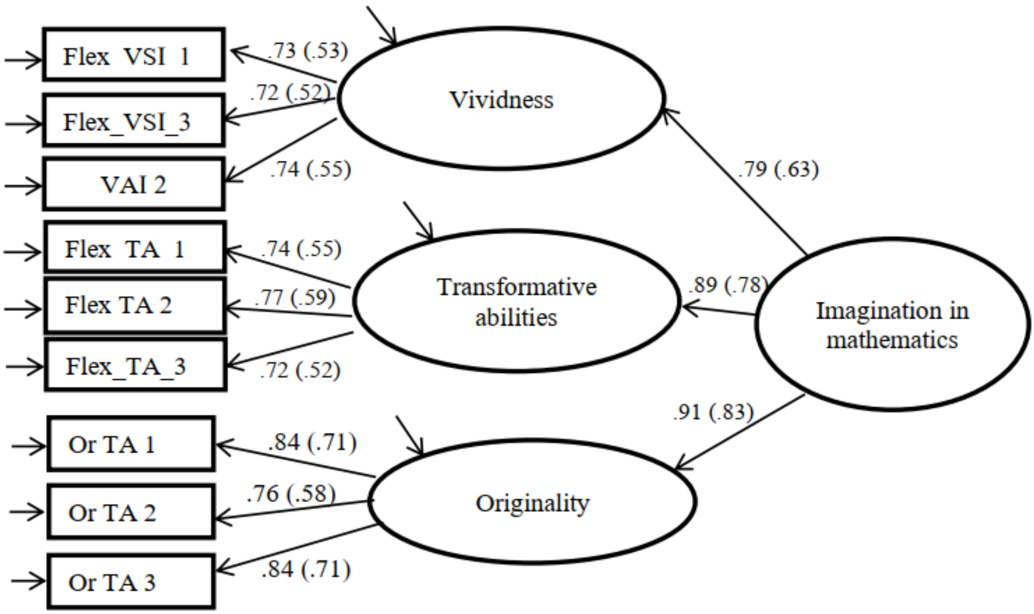
Indicators			Mean	Standard Deviation	Range	Skewness	Kurtosis
Vividness of Spatial Images (VSI)	Task 1	Flexibility	.60	.26	1	-.32	-.33
	Task 2	Flexibility	.59	.19	1	-.13	-.09
	Task 3	Flexibility	.48	.24	1	.06	-.95
Vividness of Algebraic Images (VAI)	Task 1	Correctness	.44	.40	1	.18	-1.48
	Task 2	Correctness	.62	.32	1	-.15	-1.12
	Task 3	Correctness	.51	.40	1	.03	-1.54
Transformative Abilities (TA)	Task 1	Flexibility	.45	.21	1	-.21	-.25
	Task 2	Flexibility	.26	.25	1	.48	-.82
	Task 3	Flexibility	.50	.25	1	-.22	-.33
Originality	VSI 1	Originality	.43	.23	1	-.06	-.61
	VSI 2	Originality	.27	.18	1	.98	1.51
	VSI 3	Originality	.35	.27	1	.47	-1.07
	TA 1	Originality	.34	.18	1	-.19	-.25
	TA 2	Originality	.17	.20	1	1.46	2.21
	TA 3	Originality	.34	.17	1	-.42	.39

Figure 3 shows model parameter estimates (indicators' loadings, constructs' loadings, and variance explained by the constructs). Overall, all indicators are suitable measures of the relevant constructs. As for indicators' reliability, all indicators with loadings below .70 were removed. All remaining indicators have positive, high, and significant loadings (above .70). Two spatial images' indicators and one algebraic images' indicators load strongly on the factor of vividness. In addition, all three transformative abilities' indicators exhibit satisfactory loadings on transformative abilities. Finally, only algebraic images' indicators have adequate loadings on the factor of originality, while spatial images' indicators have lower loadings.

Furthermore, the positive, high, and statistically significant loadings of the first-order factors, namely vividness (λ=.79, R^2=.63), transformative abilities (λ=.89, R^2=.78) and originality (λ=.91, R^2=.83), indicated that these abilities constitute a second-order construct, that of mathematical imagination. Among the three abilities, the most reliable ability for measuring imagination in mathematics is originality, due to its higher loading.

Figure 3

The proposed model capturing imagination in mathematics



Note: The first number represents indicators' loadings and numbers in parentheses represent the items' variance explained by the constructs (R^2).

Table 3 summarizes the evaluation criteria outcomes (Composite reliability ρ_c, Cronbach α, AVE and Q^2). Regarding the constructs' internal consistency, composite reliability ρ_c and Cronbach α met the threshold of .60, except of Cronbach α of vividness which is slightly lower than .60. A possible interpretation is that two spatial images' indicators as well as one algebraic images' indicators load strongly on the factor of vividness. Further, all AVE values were above .50. Therefore, AVE values across all items associated with a particular construct provided evidence for the construct's convergent validity.

Discriminant validity is regarded as acceptable for all constructs of the model. Concerning the Fornell-Larcker (1981) criterion, the square roots of the AVE of all constructs were greater than the correlations between the constructs of the model. Table 4 illustrates the correlation coefficients of the three constructs of the mathematical imagination model. It is revealed that all correlations among the three constructs are positive and statistically significant. The correlation between vividness and transformative abilities and correlation between vividness and originality are moderate, while the correlation between originality and transformative abilities is considered as substantial (Best & Kahn, 2007, as cited in Wonu et al., 2018). Regarding the item-level discriminant validity, each indicator loading was greater than all of its cross-loadings.

Table 3
Evaluation criteria outcomes of the reflective measurement model defining imagination in mathematics

Criterion	Vividness	Transf. Abilities	Originality	Imagination
Composite Reliability ρ_c	.78	.79	.85	.87
Cronbach α	.57	.60	.74	.84
Average Variance Extracted (AVE)	.54	.56	.66	.75

Table 4
Correlations among the constructs of the mathematics imagination model

Construct	Vividness	Transformative Abilities	Originality
Vividness	1	.56*	.57*
Transformative Abilities	.56*	1	.73*
Originality	.57*	.73*	1

Note: *Statistically significant at α=.05

4. Discussion

Imagination is a complex construct (Egan, 1992; van Alphen, 2011) with a vague definition (Ho et al., 2013). At the same time, empirical studies revolving around imagination are scarce (Egan, 1992), despite the need raised by researchers to develop theoretical models that describe imagination (Abrahamson, 2006; Dziedziewicz & Karwowski, 2015; Egan, 1992). Therefore, aiming to bridge this gap, the present paper attempts to empirically investigate the construct of imagination in mathematics. To fulfill this goal, a proposed model was established, by adapting the Conjunctional Model of Creative Imaging Ability (Dziedziewicz & Karwowski, 2015) originating from psychology to the field of mathematics.

The data of the study empirically support the structure of the proposed model, since the proposed model meets the guidelines for evaluating of PLS-SEM results. In sum, the findings illustrate that mathematical imagination is a multidimensional construct comprising three abilities: vividness, transformative abilities and originality. Moreover, the study indicates that visualization does not relate only to geometry and trigonometry (Presmeg, 1997), but is important for algebra as well (Yerushalmy et al., 1999). Besides, it was found that only algebraic images' indicators have adequate loadings on the factor of originality, while spatial images' indicators have lower loadings. A possible interpretation is that children's originality in spatial images' tasks was influenced by another factor, such as their cognitive style (Blazhenkova et al., 2011). Finally, the high originality on imagination empirically corroborates the arguments that loading of originality is the most reliable index to measure creativity (Ervynck, 1991) and imagination is strongly related to invention and originality (White, 1990).

A range of methodological limitations can be identified in the present paper, which seem to pose fruitful directions for further exploration. First, keeping in mind that the study has only examined sixth-graders selected through convenient sampling, further studies focusing on randomly selected students of a broader age range are highly needed. What is also notable is that the study took place at a single point in time. Longitudinal studies can go a step further by investigating whether the structure of the proposed model remains fairly stable over time. Focusing on data analysis, it is useful to explore whether simpler models can describe the structure of imagination in mathematics, by collecting data from a larger sample. Kline (2013) asserts that the goal of structural equation modeling is to select a model that fits the data and is as simple as possible.

The contribution of the current paper is threefold. From a theoretical perspective, the study proposes and empirically examines a model for clarifying and gauging imagination in mathematics, considering the need of developing theoretical models on imagination (Abrahamson, 2006; Dziedziewicz & Karwowski, 2015; Egan, 1992). This model bridges the research of creativity and imagination (Jankowska & Karwowski, 2015). In sum, this study empirically confirmed that imagination is a multidimensional conceptual construct consisting of three abilities: vividness, transformative abilities, and originality.

On a methodological level, the study extends the pertinent literature by designing an instrument for assessing imagination in the field of mathematics. This instrument is theory-guided since its design is based on the components of the 'Conjunctional Model of Creative Imaging Ability'. In addition, the instrument is analytical because it clearly defines all three components of imagination. Finally, the instrument is considered suitable, because its structure was confirmed through empirical data. This instrument can be administered either for research or instructional purposes.

From a practical point of view, teacher education programs do not place particular emphasis on how to engage the imagination and also many teachers do not feel confident to fuel students' imagination (Egan & Judson, 2016). Therefore, the study can advance teachers' understanding of what mathematical imagination entails and can offer a tool for measuring imagination which, in turn, can aid them in monitoring and enhancing students' mathematical imagination.

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